Math 206B Lecture 26 Notes

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1 D-Finite and Polynomially Recursive Series

Note: Today's lecture is a guest lecture. The lecture material is from section 6.4 of Stanley's Enumerative Combinatorics (Volume 2).

1.1 D-finite series

Let K be a field. We have the ring of formal power series $K[\![x]\!]$.

Definition 1.1. If $u = \sum_n f(n)x^n$, then the **formal derivative** is $\frac{d}{dx}u = u' = \sum_n nf(n)x^{n-1}$.

Proposition 1.1. *Let $u \in K[\![x]\!]$. The following are equivalent.*

1. $\dim_{K(x)}(K(x)u + K(x)u' + K(x)u'' + \cdots) < \infty$.
2. *There are $p_0, \dots, p_\ell \in K[x]$ with $p_\ell \neq 0$ such that*
$$p_d u^{(d)} + p_{d-1}u^{(d-1)} + \cdots + p_1 u' + p_0 u = 0.$$
3. *There are $q_0, \dots, q_m, q \in K[x]$ with $q_m \neq 0$ such that*
$$q_m u^{(m)} + q_{m-1}u^{(m-1)} + \cdots + q_1 u' + q_0 u = q.$$

Proof. (1) $\implies$ (2): Suppose $\dim = d$. Then $u, u', \dots, u^{(d)}$ are linearly dependent over $K(x)$. Write down the dependence relation, and clear the denominators to get the p_i.

(2) $\implies$ (3): This is a special case.

(3) $\implies$ (2): Suppose that $\deg_x(q(x)) = t \geq 0$. Differentiate the polynomial relation $t+1$ times to get a homogeneous relation involving the derivatives of u. We get $p_d = q_m \neq 0$. Solve for $u^{(d)}$ to get
$$u^{(d)} \in K(x)u + K(x)u' + \cdots + K(x)u^{(d-1)}.$$
Writing $u^{(d)}$ as a linear combination of $u, \dots, u^{(d-1)}$ and differentiating gives $u^{(d+1)} \in K(x)u + \cdots + K(x)u^{(d)} = K(x)u + \cdots + K(x)u^{(d-1)}$. By induction, for all $k \geq 0$, $u^{(d+k)} \in K(x)u + \cdots + K(x)u^{(d-1)}$. $\square$

This allows us to make the following definition.

Definition 1.2. $u \in K[\![x]\!]$ is D**-finite** if any of these three conditions hold for u.

Example 1.1. $u = e^x = \sum_n x^n/n!$ is D-finite. This satisfies $u' = u$, so $u' - u = 0$. In general, $u = x^m e^{ax}$ is D-finite, as $u' = mx^{m-1}e^{ax} + ax^m e^{ax} = (m/x + a)u$.

Example 1.2. Suppose $u = \sum_{n \geq 0} n! x^n$. Then $(xu)' = \sum_{n \geq 0}(n+1)! x^n$, so we see that $1 + x(xu)' = u$. That is, $x^2 u' + (x-1)u = -1$, so u is D-finite.

1.2 Polynomially recursive series

Definition 1.3. Say $f : \mathbb{N} \to K$ is P**-recursive** (or **polynomially recursive**) if there are $P_0, \ldots, P_e \in K[x]$ with $P_e \neq 0$ such that

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \cdots + P_0(n)f(n) = 0$$

for all $n \in \mathbb{N}$.

Proposition 1.2. *Let $u = \sum_{n \geq 0} f(n)x^n \in K[\![x]\!]$. Then u is D-finite if and only if f is P-recursive.*

Proof. First, suppose u is D-finite. Then we have polynomials p_I such that

$$p_e u^{(d)} + \cdots + p_1 u' + p_0 u = 0.$$

Then $x^j u^{(i)} = \sum_{n \geq 0}(u+i-j)_i f(u+i-j)x^n$, where $(u+i-j)_i$ is the falling factorial. For $k \gg 0$, equate coefficients of x^{n+k} in the polynomial relation to get the recurrence. Note that if $[x^j]p_d(x) \neq 0$, then $[n^d]P_{d-j+k}(n) \neq 0$ (where brackets denote the coefficient of the term inside).

Now suppose f is P-recursive. Then f satisfies a relation

$$P_e(n)f(n+e) + P_{e-1}(n)f(n+e-1) + \cdots + P_0(n)f(n) = 0$$

with $P_e \neq 0$. For each i, $\{(n+i)_j : j \geq 0\}$ is a basis for the K-vector space $K[u]$. So $P_i(n)$ is a K-linear combination of $(n+i)_j$s. So $\sum_n P_i(n)f(n+i)x^n$ is a K-linear combination of series of the form $\sum_{n \geq 0}(n+i)_j f(n+i)x^n$. Now $\sum_{n \geq 0}(n+i)_j f(n+i)x^n = R_i(x) + x^{j-i}u^{(j)}$, where $R_i \in x^{-1}K[x^{-1}]$. For example, $x^{-1}u' = \sum_{n \geq -1}(n+2)f(n+2)x^n = f(1)x^{-1} + \sum_{n \geq 0}(n+2)f(n+2)x^n$. Now multiply the relation by x^n and sum over $n \geq 0$ to get

$$0 = \sum a_{i,j} x^{j-i} u^{(j)} + R(x).$$

This sum is finite, $a_{i,j} \in K$ are not all 0, and $R(x) \in x^{-1}K[x^{-1}]$. Multiply by x^q with $q \gg 0$. We get that u is D-finite. $\square$

Example 1.3. Let $u = e^x$. Another way to show that u is D-finite is to show that $f(n) = n!$ is P-recursive: $f(n+1) - (n+1)f(n) = 0$.

Example 1.4. Let $f(n) = \binom{2n}{n}$. This is P-recursive: $(n+1)f(n+1) - 2(2n+1)f(n) = 0$. So $u = \sum_{n\geq 0} \binom{2n}{n} x^n$ is D-finite. This is the series for $1/\sqrt{1-4x}$.

Proposition 1.3. *Suppose $f, g : \mathbb{N} \to K$ is P-recursive and $f(n) = g(n)$ for all sufficiently large n. Then g is P-recursive.*

Proof. Suppose $f(n) = g(n)$ for $n \geq n_0$. Then

$$\left(\prod_{j=0}^{n_0-1} (n-j)\right) [P_e(n)g(n+e) + \cdots + P_0(n)g(n)] = 0,$$

so g is P-recursive. □

Theorem 1.1. *Suppose $u \in K[\![x]\!]$ is algebraic over $K(x)$ of degree d. Then u is D-finite and satisfies a polynomial equation of order d.*

Proof. Let $P(Y) \in K(x)[Y]$ be the minimal polynomial of u over $K(x)$. Suppose $P(Y) = \sum_{i=0}^{d} p_i Y^i$. Then $P(u) = 0$. Differentiate to get

$$0 = (P(u))' = \left(\sum_{i=0}^{d} p_i u^i\right)' = \underbrace{\sum_{i=0}^{d} p_i' u^i}_{=Q(u)} + \underbrace{\sum_{i=0}^{d} i p_i u^{i-1} u'}_{=(\frac{\partial P}{\partial Y}(u))u'}$$

The derivarive $\frac{\partial P}{\partial Y}(u) \neq 0$. So $u' = Q(u)/(\frac{\partial P}{\partial Y}(u)) \in K(x)(u)$. Similarly, $u^{(n)} \in K(x)(u)$ for all N. Then we get linear dependence among $u, u', \ldots, u^{(d)}$. □